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Two Points Blow-up in Solutions of the Nonlinear Schrödinger Equation with Quartic Potential on $\mathbb{R}$

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We consider the blow-up problem for the nonlinear Schrödinger equation with quartic self-interacting potential on $\mathbb{R}$. We exhibit a class of initial data leading to the blow-up solutions which have at least two L^2-concentration points.

KEY WORDS: Nonlinear Schrödinger equation; 1 + 1 space-time; quartic self-interacting potential; critical case; blow-up; two L^2-concentration points; variational problem.

1. INTRODUCTION AND MAIN RESULTS

This paper concerns the nonlinear Schrödinger equation with quartic self-interacting potential on $\mathbb{R}$:

$$2i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + |u|^4 u = 0, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \tag{1.1}$$

We associate this equation with initial data from the usual Sobolev space $H^1(\mathbb{R})$:

$$u(0, x) = u_0 \in H^1(\mathbb{R}), \qquad x \in \mathbb{R} \tag{1.2}$$

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As noted in ref. 5, the study of this equation may bring us insight into the behavior of solutions of the $2+1$ space-time nonlinear Schrödinger equation with nonlinear term $|u|^2 u$, which is the well-known model of a laser field propagation in a nonlinear medium (see, e.g., ref. 22, and see also refs. 1 and 6 and references therein). In fact, these two equations have common mathematical properties; the exponent in the nonlinear term called *critical* (see, e.g., refs. 2, 9, 18, and 19).

In what follows, we shall use the following notations: $D \equiv d/dx$ and $\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)}\, g(x)\, dx$, where $\overline{f(x)}$ means the complex conjugate of $f(x)$. We also abuse the notation that D is also used to denote the partial derivative $\partial/\partial x$.

We summarize here the basic properties of this Cauchy problem (1.1)–(1.2) (see, e.g., refs. 3 and 4). The unique local existence of solutions is well known: for any $u_0 \in H^1(\mathbb{R})$, there exists a unique solution $u(t, x)$ in $C([0, T_m); H^1(\mathbb{R}))$ for some $T_m \in (0, \infty]$ (maximal existence time; for simplicity, we shall consider the forward problem only), and $u(t)$ satisfies the following three conservation laws of L^2, the energy E and the momentum P in this order:

$$N(u(t)) \equiv \|u(t)\|^2 = \|u_0\|^2 \tag{1.3}$$

$$E(u(t)) \equiv \|Du(t)\|^2 - \tfrac{1}{3}\|u(t)\|_6^6 = E(u_0) \tag{1.4}$$

$$P(u(t)) \equiv \Im \int_{\mathbb{R}} u(t, x)\, D\overline{u(t, x)}\, dx = P(u_0) \tag{1.5}$$

for $t \in [0, T_m)$, where $\|\cdot\|$ and $\|\cdot\|_q$ denotes the L^2 norm and L^q norm respectively. Furthermore we have the following alternatives: $T_m = \infty$ or $T_m < \infty$ and $\lim_{t \to T_m} \|\nabla u(t)\| = \infty$ (blow-up). For the existence of blow-up solutions, see refs. 2, 12, 15, 16, and 18.

Our purpose in this paper is to exhibit a class of initial data leading to the blow-up solution to (1.1)–(1.2) which have, at least, two L^2-concentration points.

In order to state our result precisely, we need an auxiliary $W^{3,\infty}(\mathbb{R})$ odd function; we here introduce it following refs. 15 and 16, as follows:

$$\phi(\xi) = \begin{cases} \xi, & 0 \leqq \xi < 1 \\ \xi - (\xi - 1)^3, & 1 \leqq \xi < 1 + 1/\sqrt{3} \\ \text{smooth}, \ (\phi' \leqq 0) & 1 + 1/\sqrt{3} \leqq \xi < 2 \\ 0, & 2 \leqq \xi \end{cases} \tag{1.6}$$

Using this $\phi(\xi)$, we put

$$\Psi_{R,a}(x) = \phi_R(x-a) + \phi_R(x+a) = R\phi\left(\frac{x-a}{R}\right) + R\phi\left(\frac{x+a}{R}\right) \tag{1.7}$$

$$\Phi_{R,a}(x) = 2\int_0^x \Psi_{R,a}(s)\,ds + 2\int_0^{2R} \phi_R(s)\,ds \tag{1.8}$$

for given $a>0$ and $R>0$ such that

$$a - 2R > 0 \tag{1.9}$$

where $\phi_R(x) = R\phi(x/R)$. Notice that

$$\operatorname{supp} \phi_R(\cdot - a) \cap \operatorname{supp} \phi_R(\cdot + a) = \varnothing \tag{1.10}$$

and $\Phi_{R,a} = (x \pm a)^2$ if $|x \pm a| < R$. One can easily verify that

$$|D^l \Phi_{R,a}(x)| \leqq \frac{K_l}{R^{l-1}}, \qquad l = 0, 1, 2, 3 \tag{1.11}$$

$$D\Phi_{R,a}(x) = 2\Psi_{R,a}(x) \tag{1.12}$$

$$\Psi_{R,a}{}^2 \leqq \Phi_{R,a} \tag{1.13}$$

$$R^2 \leqq \Phi_{R,a}(x) < 4R^2 \qquad \text{for} \quad x \in \operatorname{supp}(1 - D\Psi_{R,a}) \tag{1.14}$$

We put

$$\varrho_{R,a}(x) \equiv 1 - D\Psi_{R,a}(x) \geqq 0 \tag{1.15}$$

$$\begin{aligned} \Omega_{R,a} &\equiv \operatorname{supp}(1 - D\Psi_{R,a}) \\ &= (-\infty, -a-R] \cup [-a+R, a-R] \cup [a+R, +\infty) \end{aligned} \tag{1.16}$$

We suppose that the initial datum $u_0 \in H^1(\mathbb{R})$ satisfies the following conditions:

(a) u_0 is an odd or even function.

(b) u_0 has, at least, negative energy:

$$-E^* \equiv E(u_0) < 0$$

(c) For some $\eta > 0$, we have

$$-\eta + \tfrac{2}{3}\|u_0\|^6 \|D\sqrt{\varrho_{R,a}}\|_\infty^2 < 0$$

(d) The energy is sufficiently negative:

$$-\eta_0 \equiv -E^* + \frac{K_3}{4R^2}\|u_0\|^2 + \eta < 0$$

Although the condition (b) is included in (d), we dare to state it for latter convenience. We note that these inequalities in (c) and (d) persist along the time evolution, $u(t)$, by the L^2 conservation law (1.3) and the energy conservation law (1.4).

We need one more condition (e) below. To state it, we introduce here the following two variational values N_c and m_R. The first one is the well-known critical L^2 norm[19] (see Remark 1.1 below):

$$N_c \equiv \inf_{\substack{v \in H^1(\mathbb{R}) \\ v \not\equiv 0}} \{ \|v\|^2 \mid E(v) \leqq 0 \} \tag{1.17}$$

The second one is its local analogue[12] (see also ref. 14):

$$m_R \equiv \inf_{\substack{v \in H^1(\mathbb{R}) \\ v \not\equiv 0}} \left\{ \int_{\Omega_{R,a}} |v(x)|^2 \, dx \;\middle|\; E_a^R(v) \leqq -\eta, \tfrac{2}{3}\|v\|^6 \|D\sqrt{\varrho_{R,a}}\|_\infty^2 < \eta \right\} \tag{1.18}$$

where

$$E_a^R(v(\cdot)) \equiv \int_{\mathbb{R}} \varrho_{R,a}(x)(|Dv(x)|^2 - \tfrac{1}{3}|v(x)|^6)\, dx \tag{1.19}$$

Remark 1.1. (1) The variational value N_c is also characterized as follows:

$$\frac{1}{3} N_c^2 = \inf_{\substack{v \in H^1(\mathbb{R}) \\ v \not\equiv 0}} \frac{\|v\|^4 \|Dv\|^2}{\|v\|_6^6} \tag{1.20}$$

and we know that there exists a positive function $Q(x) \in H^1(\mathbb{R})$ such that

$$D^2 Q - Q + |Q|^4 Q = 0 \tag{1.21}$$

$$\|Q\|^2 = N_c \tag{1.22}$$

For this, see ref. 19 (see also ref. 11). By (1.20) and the conservation laws (1.3) and (1.4), if $\|u_0\|^2 < N_c$, the solution of (1.1)–(1.2) exists globally in time. Furthermore, now we know that every blow-up solution concentrates its L^2 mass which, at least, amounts to N_c at some point (see refs. 8, 9, 20, and 21).

(2) Such a kind of variational problem as (1.18) was first introduced in ref. 12 (see also ref. 14) to prove the nonexistence of negative energy global-in-time solution. The problem (1.18) is a variation of the one introduced there. In fact the analysis of this paper is closely related to refs. 12–14 and also refs. 9 and 10. We can show as in a similar way to the proof of Lemma 3.1; ref. 12 (see also Section 2; ref. 14) that

$$m_R^2 \geqq \tfrac{3}{8} \tag{1.23}$$

Actually, this estimate from below enable us to construct the blow-up solutions which we want. Unfortunately, we do not know whether a similar estimate holds for higher dimensional critical case except the radially symmetric case (see ref. 12). We shall prove (1.23) in Section 2 as Lemma 2.1.

Now we can state the condition (e):

(e) The initial datum u_0 is localized around the support of $\Psi_{R,a}$:

$$\frac{1}{R^2}\left(\frac{1}{\eta_0}\|Du_0\|^2+1\right)\langle \Phi_{R,a}, |u_0|^2\rangle < \min(m_R, N_c)$$

Our main results are the following Theorems 1 and 2:

Theorem 1. Let $a>0$ and $R>0$ be arbitrary such that we have (1.9). Under the conditions (a)–(e), the corresponding solution $u(t)$ of (1.1)–(1.2) blows up in a finite time, and it concentrates its L^2 mass at, at least, two points. They are located in the intervals $[-a-R, -a+R]$ and $[a-R, a+R]$. More precisely, defining the following quantity A:

$$A \equiv \sup_{r>0}\left(\liminf_{t\uparrow T_m}\left(\sup_{y\in\mathbb{R}}\int_{|x-y|\leqq r/\|u(t)\|_6^3}|u(t,x)|^2\,dx\right)\right) \tag{1.24}$$

we have

$$A \geqq N_c \tag{1.25}$$

and we have that, for any $\varepsilon\in(0,1)$, there are constant $r_0>0$ and $t_0>0$ such that, for any $t\in[t_0, T_m)$ and for any $r\geqq r_0$, it holds that

$$\int_{|x\mp y(t)|\leqq r/\|u(t)\|_6^3}|u(t,x)|^2\,dx \geqq (1-\varepsilon)\,A \tag{1.26}$$

for some

$$y(t)\in[a-R, a+R] \tag{1.27}$$

Remark 1.2. The quantity A was introduced in ref. 8 to study the L^2 concentration phenomena of blow-up solutions. We may say that A measures the largest singularity. The assertion of (1.25) was already proved (for any dimensional critical case) in refs. 8 and 9. By the proof of ref. 10, we can see that $y(\cdot)$ can be taken as a right-continuous function in $t \in [t_0, T_m)$. Moreover, in our case, if $A > \frac{1}{4}\|u_0\|^2$, then we can take $y(\cdot)$ as a continuous function. Especially, if $\|u(0)\|^2 < 4N_c$, then $y(\cdot)$ is continuous.

Theorem 2 tells us that the L^2 norm amounting to $A(\geqq N_c)$ is going to concentrate around each point $y(t)$ and $-y(t)$. We can state more precisely the limiting profile of the blow-up solution:

Theorem 2. Let $a>0$ and $R>0$ be arbitrary such that we have (1.9). Under the conditions (a)–(e), the corresponding solution $u(t)$ of (1.1)–(1.2) blows up in a finite time, and it concentrates its L^2 mass at, at least, two points. They are located in the intervals $[-a-R, -a+R]$ and $[a-R, a+R]$. More precisely, there exist:

(i) a time sequence $\{s_n\}$ such that $s_n \uparrow T_m$ as $n \to \infty$;

(ii) a family of finite L-points $\{p_j\}_{j=1}^L \subset \mathbb{R}_+$ such that $|p_j - a| \leqq R$ $(j = 1, 2,..., L)$;

(iii) a family of positive constants $\{C_j\}_{j=1}^L$ such that $C_j \geqq N_c$ $(j = 1, 2,..., L)$;

(iv) a positive measure $\mu \in \mathscr{B}'$, where $\mathscr{B}'$ is the dual of $\mathscr{B} \equiv C(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

for which we have, as $n \to \infty$,

$$|u(s_n, x)|^2 \, dx \rightharpoonup \sum_{j=1}^{L} C_j(\delta_{p_j}(dx) + \delta_{-p_j}(dx)) + \mu(dx) \tag{1.28}$$

in the weak topology of measures, that is, weakly* in $\mathscr{B}'$, where δ_p is the Dirac measure at $p \in \mathbb{R}$. Furthermore we have:

$$|x| \, u_0 \notin L^2(\mathbb{R}) \Rightarrow \int_{\mathbb{R}} |x|^2 \, \mu(dx) = \infty \tag{1.29}$$

The formula (1.28) says that there are, at least, two L^2-concentration points p_1 and $-p_1$. The measure μ in (1.28) may have a singular part. However, it is considered to be the limit of the part called "shoulder" in the blow-up solution, which is of different nature from the other part producing the Dirac measures in (1.28). For the precise description of the asymptotic behavior of it, see Theorem 2.4 and Corollary 2.5 in Section 2. The assertion

(1.29) is proved in refs. 13 and 14, which shows that μ is not a finite sum of Dirac measures. We note that if $\|u_0\|^2 < 4N_c$, we have $L = 2$ in Theorem 2.

It is worth while stating the following remarks:

Remark 1.3. (1) We note that the set of initial data satisfying the conditions (a)–(e) is, of course, not empty. Let $f(x)$ be any $C_0^\infty(\mathbb{R})$ function whose support is included in the ball $\{x \in \mathbb{R}: |x| < R\}$. If necessary, multiplying it by a positive constant, we may assume that $E(f) < 0$. Now we consider the scaled function $f_\lambda(x) \equiv \lambda^{1/2} f(\lambda x)$ for $\lambda > 0$, and put $u_0(x) \equiv f_\lambda(x-a) + f_\lambda(x+a)$. First, we take $\eta > 0$ so that we have the condition (c). Note the following facts: $\|u_0\|_q^q = 2\,\|f_\lambda\|_q^q\,(1 \leqq q < \infty)$, $E(u_0) = 2\lambda^2 E(f)$. Especially, $\|u_0\|^2 = 2\,\|f\|^2$. Choosing λ large enough and replacing η by $\lambda^2\eta$, we have both the conditions (d) and (c). By this choice of η, we can assume $(1/\eta_0)\,\|\nabla u_0\|^2$ is bounded for $\lambda > 0$. Hence, if necessary, taking λ large further, we have the condition (e). Consequently, we have obtained the desired initial datum. We can add some function $g \in H^1(\mathbb{R})$ of positive energy such that

$$\operatorname{supp} g \cap \operatorname{supp}(f_\lambda(\cdot - a) + f_\lambda(\cdot + a)) = \varnothing \tag{1.30}$$

to $f_\lambda(x-a) + f_\lambda(x+a)$ without destroying the conditions (c), (d) and (e), taking $R > 0$ large enough (if necessary).

(2) In the above construction of u_0, we can replace $f \in C_0^\infty(\mathbb{R})$ by a rapidly decreasing C^∞ functions such as Q. For the properties of Q, see, e.g., ref. 19. Some numerical computations treat such an initial datum as $u_0(x) = \sum_{j=1}^{L} c_j Q(x + a_j)$ for some constants $c_j > 0$ (see, e.g., refs. 1 and 4).

(3) In ref. 7, the blow-up solution which has exactly k-given blow-up (L^2 concentration) points is constructed. But the argument there is like a construction of the "wave operators," and does not specify the initial data.

For the latter use, we define $\Psi_{R,a}$, $\Phi_{R,a}$, $\varrho_{R,a}$ for $a = 0$. In this case, we put

$$\Psi_{R,0}(x) = \phi_R = R\phi\left(\frac{x}{R}\right) \tag{1.31}$$

$$\Phi_{R,0}(x) = 2\int_0^x \phi_R(s)\,ds \tag{1.32}$$

For these newly defined functions, we also have (1.11)–(1.14) with $a = 0$. Notice that

$$\Omega_{R,0} = \{x \in \mathbb{R} \mid |x| \geqq R\} \tag{1.33}$$

This paper is organized as follows.
In Section 2, we prove Theorems 1 and 2.
In Section 3, we consider a generalization of Theorems 1 and 2.

2. PROOF OF THEOREMS 1 AND 2

Our proof of the nonexistence-of-global-solution part is a variant of Section 3, ref. 12. The important thing is to introduce the variational value (1.23) which owes its inception to ref. 12. For the formation-of-singularities part, we shall use the result of refs. 9 and 10.

Although we already know that, under the condition (b), the corresponding solution blows up in a finite time T_m (see refs. 12 and 16), our argument here includes the proof of $T_m < \infty$.

We begin with the proof of (1.23):

Lemma 2.1. Consider the following variational value m_R:

$$m_R \equiv \inf_{\substack{v \in H^1(\mathbb{R}) \\ v \not\equiv 0}} \left\{ \int_{\Omega_{R,a}} |v(x)|^2 \, dx \;\middle|\; E_a^R(v) \leqq -\eta, \tfrac{2}{3} \|v\|^6 \, \|D\sqrt{\varrho_{R,a}}\|_\infty^2 < \eta \right\} \tag{2.1}$$

Then we have

$$m_R^2 \geqq \tfrac{3}{8} \tag{2.2}$$

Proof. As mentioned in Section 1, the proof here is almost the same as in Lemma 3.1, ref. 12. For the readers convenience, we give here the proof of it. First note that $D\sqrt{\varrho_{R,a}} \in L^\infty(\mathbb{R})$: Indeed; since we have by (1.6) that

$$|D\sqrt{\varrho_{R,a}}(x)| \leqq \begin{cases} 0 & 0 \leqq |x \pm a| < R \\ \sqrt{3}\, D\left|\dfrac{x}{R} - 1\right|, & R \leqq |x \pm a| < R\left(1 + \dfrac{1}{\sqrt{3}}\right) \\ \dfrac{1}{2}\varrho_{R,a}{}^{-1/2}\, |D^2\Psi_{R,a}(x)|, & R\left(1+\dfrac{1}{\sqrt{3}}\right) \leqq |x \pm a| < 2R \\ 0, & a + 2R \leqq |x|, \ |x| \leqq a - 2R \end{cases}$$

and that

$$\varrho_{R,a}(x) = 1 - D\Psi_{R,a}(x) \geqq 1 \quad \text{on} \quad \mathbb{R} \Big\backslash \left\{ x \in \mathbb{R} \;\middle|\; |x \pm a| \leqq R\left(1 + \frac{1}{\sqrt{3}}\right) \right\} \tag{2.4}$$

we have from (1.9) that

$$|D\sqrt{\varrho_{R,a}}(x)| \leqq \frac{\max(\sqrt{3}, K_2/2)}{R} \tag{2.5}$$

Recall the weighted interpolation inequality (see, e.g., refs. 12 and 16),

$$\int_{\mathbb{R}} \varrho_{R,a}(x)\, |v(x)|^6\, dx \leqq 2\, \|v\|^2_{L^2(\Omega_{R,a})}\, \|D\sqrt{\varrho_{R,a}}\|^2_\infty + 8\, \|v\|^4_{L^2(\Omega_{R,a})}\, \|\sqrt{\varrho_{R,a}}\, Dv\|^2 \tag{2.6}$$

From (2.6) and the definition of E_a^R, we get

$$\eta + \int_{\Omega_{R,a}} \varrho_{R,a}\, |Dv(x)|^2\, dx \leqq \tfrac{8}{3}\, \|v\|^4_{L^2(\Omega_{R,a})} \int_{\Omega_{R,a}} \varrho_{R,a}\, |Dv(x)|^2\, dx + \tfrac{2}{3}\, \|v\|^6_{L^2(\Omega_{R,a})}\, \|D\sqrt{\varrho_{R,a}}\|^2_\infty \tag{2.7}$$

Consequently we have, by the condition (c), that

$$\left(1 - \tfrac{8}{3}\, \|v\|^4_{L^2(\Omega_{R,a})}\right) \|\sqrt{\varrho_{R,a}}\, Dv\|^2 \leqq -\eta + \tfrac{2}{3}\, \|u_0\|^6_{L^2(\Omega_{R,a})}\, \|D\sqrt{\varrho_{R,a}}\|^2_\infty < 0 \tag{2.8}$$

so that we obtain

$$\tfrac{3}{8} \leqq \|v\|^4_{L^2(\Omega_{R,a})} \tag{2.9}$$

Nonexistence of global-in-time solution. We shall prove that if the initial datum u_0 satisfies conditions (c), (d) and (e), the solution of (1.1)–(1.2) blows up in a finite time. The following is a key identity of our proof here, which is a substitute of the virial identity used in refs. 2, 18, and 22.

Lemma 2.2. We assume (1.19). The solution $u(t)$ of (1.1)–(1.2) satisfies that, for $t \in [0, T_m)$,

$$\begin{aligned}\langle \Phi_{R,a}, |u(t)|^2\rangle &= \langle \Phi_{R,a}, |u_0|^2\rangle + 2\mathfrak{I}t\langle u_0, \Psi_{R,a} Du_0\rangle - t^2 E^* \\ &\quad - 2\int_0^t ds \int_0^s d\tau\, E_a^R(u(\tau)) \\ &\quad - \tfrac{1}{2}\int_0^t ds \int_0^s d\tau \langle D^3\Phi_{R,a}, |u(\tau)|^2\rangle \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}\end{aligned} \tag{2.10}$$

This identity also holds true for the case of $a=0$ with $\Psi_{R,0}$ and $\Phi_{R,0}$ defined in (1.31) and (1.32) respectively.

For the proof, see refs. 12 and 16.

Using (2.10), we can prove:

Lemma 2.3. We assume (c), (d) and (e). Let

$$T_0 \equiv \sup \left\{ t \in [0, T_m) \,\middle|\, \int_{\Omega_{R,a}} |u(\tau, x)|^2 \, dx < \min(m_R, N_c),\ 0 \leqq \tau < t \right\} \tag{2.11}$$

Then we have

$$T_0 = T_m \tag{2.12}$$

Proof. Suppose that $T_0 < T_m$. By the L^2-continuity, we have $\int_{\Omega_{R,a}} |u(T_0, x)|^2 \, dx = \min(m_R, N_c)$. By the definition of m_R and the H^1-continuity, we have

$$-\eta \leqq E_a^R(u(t)) \quad \text{for} \quad t \in [0, T_0] \tag{2.13}$$

Thus it follows that

$$\mathrm{II} = -2 \int_0^t ds \int_0^s d\tau E_a^R(u(\tau)) \leqq \eta t^2 \quad \text{for} \quad t \in [0, T_0] \tag{2.14}$$

The third term (III) of right hand side of (2.10) can be easily handled; we have by (1.11) that

$$|\mathrm{III}| \leqq \frac{K_3}{4R^2} \|u_0\|^2 t^2 \tag{2.15}$$

Thus, we have by the condition (d) that, for $t \in [0, T_0]$,

$$\begin{aligned}
\langle \Phi_{R,a}, |u(t)|^2 \rangle &\leqq \langle \Phi_{R,a}, |u_0|^2 \rangle + 2\Im \langle u_0, \Psi_{R,a} D u_0 \rangle\, t - \eta_0 t^2 \\
&\leqq -\eta_0 \left(t - \frac{1}{\eta_0} \Im \langle u_0, \Psi_{R,a} D u_0 \rangle \right)^2 \\
&\quad + \frac{1}{\eta_0} |\langle u_0, \Psi_{R,a} D u_0 \rangle|^2 + \langle \Phi_{R,a}, |u_0|^2 \rangle \\
&\leqq \left(\frac{1}{\eta_0} \|D u_0\|^2 + 1 \right) \langle \Phi_{R,a}, |u_0|^2 \rangle
\end{aligned} \tag{2.16}$$

In the last inequality, we have used the fact that ${\Psi_{R,a}}^2 \leqq \Phi_{R,a}$. From (2.16), we have

$$\int_{\mathbb{R}} \Phi_{R,a}(x)\, |u(t,x)|^2\, dx \leqq \left(\frac{1}{\eta_0} \|Du_0\|^2 + 1\right) \langle \Phi_{R,a}, |u_0|^2\rangle, \quad t \in [0, T_0] \tag{2.17}$$

Dividing both sides of (2.16) by R^2, we obtain from the condition (e) that

$$\begin{aligned}\int_{\Omega_{R,a}} |u(t,x)|^2\, dx &\leqq \frac{1}{R^2}\left(\frac{1}{\eta_0} \|Du_0\|^2 + 1\right) \langle \Phi_{R,a}, |u_0|^2\rangle \\ &< \min(m_R, N_c), \quad t \in [0, T_0]\end{aligned} \tag{2.18}$$

which contradicts the definition of T_0.

This lemma tells us that the solution $u(t)$ in consideration is well-localized around the support of $\Psi_{R,a}$, and this is the heart of the matter in the entire proofs of Theorems 1 and 2.

By the definition of m_R, we have

$$-\eta \leqq E_a^R(u(t)) \quad \text{for} \quad t \in [0, T_m) \tag{2.19}$$

so that it follows that

$$\mathrm{II} = -2\int_0^t ds \int_0^s d\tau\, E_a^R(u(\tau)) \leqq \eta t^2 \quad \text{for} \quad t \in [0, T_m) \tag{2.20}$$

Consequently, by the condition (d) (definition of η_0), we obtain

$$\langle \Phi_{R,a}, |u(t)|^2\rangle \leqq \langle \Phi_{R,a}, |u_0|^2\rangle + 2\Im\langle u_0, \Psi_{R,a} Du_0\rangle\, t - \eta_0 t^2 \tag{2.21}$$

This implies that $\langle \Phi_{R,a}, |u(t)|^2\rangle$ becomes negative in finite time. Therefore, if we suppose $T_m = \infty$, then we reach a contradiction. Thus we have proved that, under the conditions (c), (d) and (e), the solutions of (1.1)-(1.2) blows up in a finite time.

Proof of Theorem 1. Now we assume the condition (a) in addition to (c), (d) and (e). We note that if the initial datum $u_0(x)$ is even (odd), then the corresponding solution $u(t,x)$ is also even (odd) with respect to $x \in \mathbb{R}$ for each $t \in [0, T_m)$.

In refs. 8 and 9, we proved that, for every blow-up solution of (1.1), the quantity A defined by (1.24) satisfies (1.25). Therefore, noting the localization property proved in Lemma 2.3, we have (1.26) with (1.27) by the definition of A and the symmetry of the solution in consideration.

Proof of Theorem 2. Next, we shall prove Theorem 2. As in the same way to the proof of Theorem 1, ref. 11 (see also ref. 12), we can show the following:

Theorem 2.4. Suppose that u_0 satisfies condition (a) and the solution $u(t)$ of (1.1)–(1.2) blows up in a finite time (or grows up at infinity). Let $\{t_n\}$ be any sequence such that, as $n \to \infty$,

$$t_n \uparrow T_m, \qquad \sup_{t \in [0, t_n)} \|u(t)\|_6 = \|u(t_n)\|_6 \tag{2.22}$$

For this $\{t_n\}$, we put

$$\lambda_n = \frac{1}{\|u(t_n)\|_6^3} \tag{2.23}$$

and, we consider the scaled functions

$$u_n(t, x) = \lambda_n^{1/2} \overline{u(t_n - \lambda_n^2 t, \lambda_n x)} \tag{2.24}$$

for $t \in [-(T_m - t_n)/\lambda_n^2, t_n/\lambda_n^2)$. Then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), which satisfies the following properties: there exist

(i) a finite number (say L) of nontrivial solutions $u^1, u^2, ..., u^L$ of (1.1) in $\mathscr{B}(\mathbb{R}_+; H^1(\mathbb{R}))$ with

$$E(u^j) = 0 \qquad \text{and} \qquad \Im \int_{\mathbb{R}^N} Du^j(t, x)\, \overline{u^j(t, x)}\, dx = 0$$

for $j = 1, 2, ..., L$, and

(ii) sequences $\{\gamma_n^1\}, \{\gamma_n^2\}, ..., \{\gamma_n^L\}$ in $\mathbb{R}_+$ such that $\lim_{n \to \infty} |\gamma_n^j - \gamma_n^k| = \infty$ $(j \neq k)$, such that, for any $T > 0$,

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \left\| u_n(t, \cdot) - \sum_{j=1}^{L} u^j(t, \cdot - \gamma_n^j) \pm \sum_{j=1}^{L} u^j(t, \cdot + \gamma_n^j) \right\|_\sigma = 0 \tag{2.25}$$

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \left\| Du_n(t, \cdot) - \sum_{j=1}^{L} Du^j(t, \cdot - \gamma_n^j) \pm \sum_{j=1}^{L} Du^j(t, \cdot + \gamma_n^j) \right\| = 0 \tag{2.26}$$

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \left\| u_n(t, \cdot) - \sum_{j=1}^{L} u^j(t, \cdot - \gamma_n^j) \pm \sum_{j=1}^{L} u^j(t, \cdot + \gamma_n^j) - \varphi_n(t, \cdot) \right\| = 0 \tag{2.27}$$

where $\varphi_n(t, x)$ is the solution of the following free Schrödinger equation:

$$\begin{cases} 2i \dfrac{\partial \varphi_n}{\partial t} + D^2 \varphi_n = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ \varphi_n(0, x) = u_n(0, x) - \displaystyle\sum_{j=1}^{L} u^j(0, x - \gamma_n^j) \pm \sum_{j=1}^{L} u^j(0, x + \gamma_n^j), & x \in \mathbb{R} \end{cases} \tag{2.28}$$

Here, the each sign $\pm$ in front of the second $\sum$'s in (2.25)–(2.28) is taken to be $-$ $(+)$ if u_0 is even (odd). Furthermore we have

$$\|u_0\|^2 \geqq 2 \sum_{j=1}^{L} \|u^j(t)\|^2 \geqq 2LN_c \tag{2.29}$$

From this theorem, we can show (see refs. 13 and 14):

Corollary 2.5. Under the same assumptions, definitions and notations of Theorem 2.4, we have:

$$\lim_{n \to \infty} \sup_{t \in [t_n - \lambda_n^2 T, t_n]} \left\| \overline{u(t, \cdot)} - \sum_{j=1}^{L} u_n^j(t, \cdot) \pm \sum_{j=-1}^{-L} u_n^j(t, \cdot) - \tilde{\varphi}_n(t, \cdot) \right\| = 0 \tag{2.30}$$

with

$$\lim_{n \to \infty} \lambda_n^2 \sup_{t \in [0, T]} \|\tilde{\varphi}_n(t)\|_6^6 = 0 \tag{2.31}$$

where, for $j = 1, 2, \ldots, L$,

$$u_n^{\pm j}(t, x) = \frac{1}{\lambda_n^{1/2}} u^{\pm j}\left(\frac{t_n - t}{\lambda_n^2}, \frac{x \mp \gamma_n^j \lambda_n}{\lambda_n}\right) \tag{2.32}$$

$$\tilde{\varphi}_n(t, x) = \frac{1}{\lambda_n^{1/2}} \varphi_n\left(\frac{t_n - t}{\lambda_n^2}, \frac{x}{\lambda_n}\right) \tag{2.33}$$

Here, each sign above is taken to be $-$ $(+)$ if u_0 is even (odd). Furthermore we have, for any $T > 0$ and any $f \in \mathscr{B}$,

$$\lim_{n \to \infty} \sup_{t \in [t_n - \lambda_n^2 T, t_n]} \left| \int_{\mathbb{R}} \left(|u(t, x)|^2 - \sum_{j=-L}^{L} |u_n^j(t, x)|^2 - |\tilde{\varphi}_n(t, x)|^2 \right) f(x)\, dx \right| = 0 \tag{2.34}$$

and

$$|\gamma_n^j \lambda_n - a| < R \qquad (j = 1, 2, ..., L) \tag{2.35}$$

Theorem 2.4 is almost the same as Theorem 1, ref. 11. The novelty is only the fact that the "singularities" u_n^j $(j = \pm 1, \pm 2, ..., \pm L)$ appear symmetrically on $\mathbb{R}$, which is a consequence of the condition (a). The formula (2.30) is just a rephrase of (2.27). For (2.34), see ref. 13, which can be obtained by the proof of (2.27). The important thing is the assertion (2.35), which is derived from Lemma 2.3 together with the fact that $\|u^j\|^2 \geqq N_c$ which is a conclusion of the fact that $E(u^j) = 0$ (see (1.17); the definition of N_c).

On the other hand, in refs. 12 and 14, we proved the following:

Lemma 2.6. We suppose the condition (b). Then, the blow-up solution of (1.1)–(1.2) satisfies the following property: there exists $m_* > 0$ such that for any $m \in (0, m_*)$, there exists a constant $r_m > 0$ such that

$$\int_{|x| > r_m} |u_0(x)|^2 \, dx < m \Rightarrow \int_{|x| > r_m} |u(t, x)|^2 \, dx < m \qquad t \in (0, T_m) \tag{2.36}$$

Furthermore, we have:

$$\int_0^{T_m} (T_m - t) \left(\int_{|x| > r_m} |Du(t, x)|^2 \, dx \right) dt < \infty \tag{2.37}$$

$$\int_0^{T_m} (T_m - t) \left(\int_{|x| > r_m} |u(t, x)|^6 \, dx \right) dt < \infty \tag{2.38}$$

From (2.36), we see that the family of "probability measures" $\{ |u(t, x)|^2 \, dx \}_{t \geqq 0}$ is *tight.* Hence, since we have (2.35), we can show (1.25) along a subsequence of $\{s_n\}$ defined by

$$s_n = t_n - \lambda_n^2 T \tag{2.39}$$

where $\{t_n\}$ is given by (2.22). Notice that

$$u_n^{\pm j}(s_n, x) = \frac{1}{\lambda_n^{1/2}} u^{\pm j} \left(T, \frac{x \mp \gamma_n^j \lambda_n}{\lambda_n} \right) \tag{2.40}$$

$$\tilde{\varphi}_n(s_n, x) = \frac{1}{\lambda_n^{1/2}} \varphi_n \left(T, \frac{x}{\lambda_n} \right) \tag{2.41}$$

Remark 2.1. The proof of Lemma 3.1, ref. 12 (see also Theorem D, ref. 14) is the model of the argument performed in this paper; one can prove Lemma 2.6 by modifying the proofs of Lemmata 2.1 and 2.3 in this section. In that proof, we use the identity (2.10) with $a=0$ such as mentioned in Lemma 2.2.

It remains to prove (1.29), that is,

$$|x|\, u_0 \notin L^2(\mathbb{R}) \Rightarrow \int_{\mathbb{R}} |x|^2 \mu(dx) = \infty$$

Although, as mentioned in Section 1, we already proved this in refs. 13 and 14, we give the proof here. It suffices to prove the following:

$$\int_{\mathbb{R}} |x|^2 \mu(dx) < \infty \Rightarrow |x|\, u_0 \in L^2(\mathbb{R})$$

By (2.37) and (2.38), we have from (2.10) with $a=0$ that there are positive constants C_1 and C_2 such that, for any $r > r_m$,

$$\langle \Phi_{r,0}, |u(t)|^2 \rangle \geqq \langle \Phi_{r,0}, |u_0|^2 \rangle - C_1 \langle \Phi_{r,0}, |u_0|^2 \rangle^{1/2} \|Du_0\| - C_2 \qquad (2.42)$$

Suppose the contrary that $|x|\, u_0 \notin L^2(\mathbb{R})$. Then we have from (2.42) that

$$\lim_{r \to \infty} \liminf_{t \to T_m} \langle \Phi_{r,0}, |u(t)|^2 \rangle = \infty \qquad (2.43)$$

On the other hand, we have by the tightness of "probability measure" $\{|u(s_n, x)|^2\, dx\}_{n \geqq 1}$ that, for an enough subsequence of $\{s_n\}$ (still denoted by the same letter), that

$$\lim_{n \to \infty} \langle \Phi_{r,0}, |u(s_n)|^2 \rangle = \int_{\mathbb{R}^N} \Phi_{r,0}(x)\, \nu(dx) \qquad (2.44)$$

where

$$\nu(dx) = \sum_{j=1}^{L} C_j(\delta_{p_j}(dx) + \delta_{-p_j}(dx)) + \mu(dx) \qquad (2.45)$$

Taking $r \uparrow \infty$ in (2.44) leads us to a contradiction, since $\Phi_{r,0} \to |x|^2$ as $r \uparrow \infty$ and we have

$$\int_{\mathbb{R}} |x|^2\, \nu(dx) < \infty \qquad (2.46)$$

3. FURTHER RESULTS AND COMMENT

In this section, we consider the generalization of Theorems 1 and 2. The question is: Can we give a class of initial data which give rise to blow-up solutions with, at least, three L^2-concentration points? Unfortunately, we cannot give an answer to this problem. However, we have a kind of "controllability" theorem for blow-up points.

We shall consider a class of "multi lump" (say N-lump) initial data. As it will turn out below, we may call the class of initial data considered in Section 1 a set of symmetric 2-lump initial data.

Now take any N-distinct points $\mathbf{a} \equiv \{a_1, a_2, ..., a_N\}$ from $\mathbb{R}$; we assume $a_1 < a_2 < \cdots < a_N$ and $a_k = 0$ for some $k \in \{1, 2, ..., N\}$ for simplicity. We choose $R > 0$ such that

$$a_j + 2R < a_{j+1} - 2R \qquad \text{for any} \quad j = 1, 2, ..., N-1 \tag{3.1}$$

equivalently

$$4R < \min_{j=1, 2, ..., N-1} (a_{j+1} - a_j) \tag{3.2}$$

If $N = 1$, we can take $R > 0$ arbitrarily. Using $\phi(\xi)$ defined by (1.6), we put (as in Section 1), for the above $R > 0$,

$$\Psi_{R, \mathbf{a}}(x) = \sum_{j=1}^{N} \phi_R(x - a_j) = R \sum_{j=1}^{N} \phi\left(\frac{x - a_j}{R}\right) \tag{3.3}$$

$$\Phi_{R, \mathbf{a}}(x) = 2 \int_0^x \Psi_{R, \mathbf{a}}(s)\, ds \tag{3.4}$$

Notice that

$$\operatorname{supp} \phi_R(\cdot - a_j) \cap \operatorname{supp} \phi_R(\cdot - a_k) = \varnothing \quad \text{for} \quad j \neq k \tag{3.5}$$

As in (1.11)–(1.24), we have:

$$|D^l \Phi_{R, \mathbf{a}}(x)| \leqq \frac{K_l}{R^{l-1}}, \qquad l = 0, 1, 2, 3 \tag{3.6}$$

$$D\Phi_{R, \mathbf{a}}(x) = 2\Psi_{R, \mathbf{a}}(x) \tag{3.7}$$

$$\Psi_{R, \mathbf{a}}{}^2 \leqq \Phi_{R, \mathbf{a}} \tag{3.8}$$

$$R^2 \leqq \Phi_{R, \mathbf{a}}(x) < 4R^2 \quad \text{for} \quad x \in \operatorname{supp}(1 - D\Psi_{R, \mathbf{a}}) \tag{3.9}$$

We put

$$\varrho_{R,\mathbf{a}} \equiv 1 - D\Psi_{R,\mathbf{a}}(x) \geqq 0 \tag{3.10}$$

$$\begin{aligned}\Omega_{R,\mathbf{a}} &\equiv \operatorname{supp}(1 - D\Psi_{R,\mathbf{a}}) \\ &= (-\infty, a_1 - R] \cup \bigcup_{j=1}^{N-1} [a_j + R, a_{j+1} - R] \cup [a_N + R, +\infty)\end{aligned} \tag{3.11}$$

The N-lump initial datum $u_0 \in H^1(\mathbb{R})$ is defined to satisfy the following three conditions (α), (β), and (γ):

(α) For some $\eta > 0$, we have

$$-\eta + \tfrac{2}{3} \|u_0\|^6 \|D\sqrt{\varrho_{R,\mathbf{a}}}\|_\infty^2 < 0$$

(β) The energy is sufficiently negative:

$$-\eta_0 \equiv E(u_0) + \frac{K_3}{4R^2} \|u_0\|^2 + \eta < 0$$

To state the third condition corresponding to (e) in Section 1, we introduce:

$$m_{R,\mathbf{a}} \equiv \inf_{\substack{v \in H^1(\mathbb{R}) \\ v \not\equiv 0}} \left\{ \int_{\Omega_{R,\mathbf{a}}} |v(x)|^2 \, dx \;\middle|\; E_{\mathbf{a}}^R(v) \leqq -\eta, \tfrac{2}{3} \|v\|^6 \|D\sqrt{\varrho_{R,\mathbf{a}}}\|_\infty^2 < \eta \right\} \tag{3.12}$$

where

$$E_{\mathbf{a}}^R(v(\cdot)) \equiv \int_{\mathbb{R}} \varrho_{R,\mathbf{a}}(x) \left(|Dv(x)|^2 - \tfrac{1}{3} |v(x)|^6 \right) dx \tag{3.13}$$

As in the same way to prove Lemma 2.1, we obtain

$$m_{R,\mathbf{a}} \geqq \tfrac{3}{8} \tag{3.14}$$

Now we can state the third condition (γ):

(γ) The initial datum u_0 is localized around the support of $\Psi_{R,\mathbf{a}}$:

$$\frac{1}{R^2} \left(\frac{1}{\eta_0} \|Du_0\|^2 + 1 \right) \langle \Phi_{R,\mathbf{a}}, |u_0|^2 \rangle < \min(m_{R,\mathbf{a}}, N_c)$$

As in the same way to prove Theorems 1 and 2, we can show the following theorem:

Theorem 3. Under the conditions (α), (β) and (γ), the corresponding solution $u(t)$ of (1.1)–(1.2) blows up in a finite time, and it concentrates its L^2 mass, at least, in one of the intervals $[a_j - R, a_j + R]$ for $j = 1, 2, \ldots, N$. More precisely, there exist:

(i) a time sequence $\{s_n\}$ such that $s_n \uparrow T_m$ as $n \to \infty$;

(ii) a family of finite L-points $\{p_l\}_{l=1}^{L} \subset \mathbb{R}$ such that

$$\{p_1, p_2, \ldots, p_L\} \subset \bigcup_{j=1}^{N} [a_j - R, a_j + R]$$

(iii) a family of positive constants $\{C_l\}_{l=1}^{L}$ such that $C_l \geqq N_c$ $(l = 1, 2, \ldots, L)$;

(iv) a positive measure $\mu \in \mathscr{B}'$,

for which we have, as $n \to \infty$,

$$|u(s_n, x)|^2 \, dx \rightharpoonup \sum_{l=1}^{L} C_l \delta_{p_l}(dx) + \mu(dx) \tag{3.15}$$

in the weak topology of measures. We note that if $|x| \, u_0$ does not belong to $L^2(\mathbb{R})$, then we have

$$\int_{\mathbb{R}} |x|^2 \, \mu(dx) = \infty \tag{3.16}$$

Furthermore: for any $\varepsilon \in (0, 1)$, there are constant $r_0 > 0$ and $j_0 \in \{1, 2, \ldots, N\}$ such that we have, for any $r \geqq r_0$ and for some sequence $\{y_n\}$;

$$y_n \in [a_{j_0} - R, a_{j_0} + R] \tag{3.17}$$

that

$$\int_{|x - y_n| \leqq r/\|u(s_n)\|_6^3} |u(s_n, x)|^2 \, dx \geqq (1 - \varepsilon) \, N_c \tag{3.18}$$

for sufficiently large $n \in \mathbb{N}$.

Hence, taking $N = 1$ and $R > 0$ sufficiently small, we can almost control the L^2-concentration points, although the number of singularities, L, in

Theorem 3.1 may be larger than two. If we take $\mathbf{a} = \{-a, 0, a\}$ for some $a > 0$ and if u_0 is an *even* or *odd* function made from three lumps which can be constructed as in Remark 1.3, then the origin of $\mathbb{R}$ is possibly an L^2-concentration point. However, we do not know exactly which initial datum of such a shape gives rise to the blow-up solution with $0 \in \mathbb{R}$ as its L^2-concentration point.

We conclude this paper with the following remark: the arguments here work for the nonlinear Schrödinger equation (1.1) on the circle (whose blow-up problem is considered in refs. 5 and 17), so that we can prove the analogous results to Theorems 1, 2 and 3. These will be discussed in a forth coming paper.

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